Thermal Pressurization with Green's Functions

The thermal pressurization equations are:

$$\frac{\partial T}{\partial t} = \alpha_{\mathrm{th}} \frac{\partial^2 T}{\partial z^2} + \frac{\tau V}{\rho c h \sqrt{2\pi}} \exp\left(-\frac{z^2}{2h^2}\right) \tag{1}$$

$$\frac{\partial p}{\partial t} = \alpha_{\mathrm{hy}} \frac{\partial^2 p}{\partial z^2} + \Lambda \frac{\partial T}{\partial t} \tag{2}$$

These are linear equations for T and p, where τV acts as a source term. So the solution may be expressed in terms of Green's functions,

$$T(z,t) = T_{\mathrm{ini}} + \int_0^t G_{\mathrm{th}}(z, t - t')\tau(t')V(t')\,dt' \tag{3}$$

$$p(z,t) = p_{\mathrm{ini}} + \int_0^t G_{\mathrm{hy}}(z, t - t')\tau(t')V(t')\,dt' \tag{4}$$

It so happens that for the given pressurization equations, the Green's functions can be written in closed form:

$$G_{\mathrm{th}}(z,t) = \frac{1}{\rho c \sqrt{4\pi\alpha_{\mathrm{th}} t + 2\pi h^2}} \exp\left(-\frac{z^2}{4\alpha_{\mathrm{th}} t + 2h^2}\right) \tag{5}$$

$$\begin{aligned} G_{\mathrm{hy}}(z,t) = {} & \frac{\Lambda \alpha_{\mathrm{hy}}}{\rho c(\alpha_{\mathrm{hy}} - \alpha_{\mathrm{th}})\sqrt{4\pi\alpha_{\mathrm{hy}} t + 2\pi h^2}} \exp\left(-\frac{z^2}{4\alpha_{\mathrm{hy}} t + 2h^2}\right) \\ & - \frac{\Lambda \alpha_{\mathrm{th}}}{\rho c(\alpha_{\mathrm{hy}} - \alpha_{\mathrm{th}})\sqrt{4\pi\alpha_{\mathrm{th}} t + 2\pi h^2}} \exp\left(-\frac{z^2}{4\alpha_{\mathrm{th}} t + 2h^2}\right) \end{aligned} \tag{6}$$

Since $z = 0$ is the fault surface, and we are interested in the pore pressure $p(0,t)$ at the fault surface, it is convenient to write out the formula:

$$G_{\mathrm{hy}}(0,t) = \frac{\Lambda}{\rho c\left(\alpha_{\mathrm{hy}} - \alpha_{\mathrm{th}}\right) 2\sqrt{\pi}} \left[\frac{\alpha_{\mathrm{hy}}}{\sqrt{\alpha_{\mathrm{hy}} t + \frac{1}{2}h^2}} - \frac{\alpha_{\mathrm{th}}}{\sqrt{\alpha_{\mathrm{th}} t + \frac{1}{2}h^2}} \right] \tag{7}$$

The Green's function can be applied in the following way. Consider a series of times

$$0 = t_0 < t_1 < t_2 < t_3 < \cdots \tag{8}$$

Let τ_i be the average shear stress during the time interval $[t_{i-1}, t_i]$, and let V_i be the average slip rate during the time interval $[t_{i-1}, t_i]$. One way to define V_i is to take $V_i = |U_i - U_{i-1}|/(t_i - t_{i-1})$ where U_i is the slip vector at time t_i. Then, the pore pressure p_n at time t_n can be taken as:

$$p_n = p_{\text{ini}} + \sum_{i=1}^{n} G_{\text{hy}}(0, t_n - \tfrac{1}{2}(t_i + t_{i-1}))\tau_i V_i (t_i - t_{i-1}) \tag{9}$$

Equation 9 is a discretized version of equation 4. In writing equation 9, we are pretending that all the heat produced during the time interval $[t_{i-1}, t_i]$ is produced instantaneously at the midpoint of the interval. This approximation seems to be good enough in practice.

In order to apply equation 9, the computer code must store the values of $\tau_i V_i$ for all time steps since the beginning of the simulation.

Note that it is not necessary to compute the temperature, although one could easily do so:

$$T_n = T_{\text{ini}} + \sum_{i=1}^{n} G_{\text{th}}(0, t_n - \tfrac{1}{2}(t_i + t_{i-1}))\tau_i V_i (t_i - t_{i-1}) \tag{10}$$