Thermal Pressurization with Finite Differences

The thermal pressurization equations,

$$\frac{\partial T}{\partial t} = \alpha_{th}\frac{\partial^2 T}{\partial z^2} + \frac{\tau V}{\rho c h\sqrt{2\pi}}\exp\left(-\frac{z^2}{2h^2}\right), \tag{1}$$

$$\frac{\partial p}{\partial t} = \alpha_{hy}\frac{\partial^2 p}{\partial z^2} + \Lambda\frac{\partial T}{\partial t}, \tag{2}$$

can be solved with finite differences, as described below and implemented in the Matlab script thermpres_fd.m. Equations (1) and (2) are discretized on a finite interval at N+1 equally spaced grid points

$$z_i = i\Delta z,\ \ i = 0,\ldots,N. \tag{3}$$

As the physical problem is for an infinite domain, the location of the exterior boundary must chosen to be sufficiently far removed from the fault that boundary effects do not influence the solution. A rough estimate is obtained from the diffusion length $\sqrt{4\alpha t}$, where α is the larger of the two diffusivities. Second order central spatial difference operators are used:

$$\left(\frac{\partial^2 T}{\partial z^2}\right)_i \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta z^2},\ \ i = 0,\ldots,N. \tag{4}$$

This formula involves T_{-1} and T_{N+1}, which are determined by the boundary conditions as

$$\begin{aligned} \left.\frac{\partial T}{\partial z}\right|_{z=0} &\approx \frac{T_1 - T_{-1}}{2\Delta z} \Rightarrow T_{-1} \approx T_1 - 2\Delta z\left(\left.\frac{\partial T}{\partial z}\right|_{z=0}\right) \text{ and} \\ \left.\frac{\partial T}{\partial z}\right|_{z=N\Delta z} &\approx \frac{T_{N+1} - T_{N-1}}{2\Delta z} \Rightarrow T_{N+1} \approx T_{N-1} + 2\Delta z\left(\left.\frac{\partial T}{\partial z}\right|_{z=N\Delta z}\right). \end{aligned} \tag{5}$$

Time stepping can be done implicitly or explicitly. The simplest method is explicit Euler with constant time step Δt, in which an update from time $t^n = n\Delta t$ to $t^{n+1} = (n+1)\Delta t$ is accomplished by approximating time derivatives as

$$\left(\frac{\partial T}{\partial t}\right)^n \approx \frac{T^{n+1} - T^n}{\Delta t} \tag{6}$$

and evaluating the right-hand sides of (1) and (2) time step n. The expressions can be solved explicitly for the values of T and p at time step n+1. Stability requires that

$$\Delta t \le C\frac{\Delta z^2}{\alpha}, \tag{7}$$

where α is the larger of the diffusivities and C is an $O(1)$ coefficient that depends on the specific spatial difference operator and the method of enforcing the boundary conditions.

Exact solutions to the thermal pressurization equations are known only for the case of slip-on-a-plane, which is obtained in the $h \to 0$ limit, and for sliding with constant friction coefficient f and constant slip velocity V. For this case, the rate of heat production in (1) is

$$\tau V = f\left(\sigma - p\big|_{z=0}\right)V, \tag{8}$$

and the shear strength decays with slip as (Rice, *J. Geophys. Res.*, 2006)

$$\tau(\delta = Vt) = f\left(\sigma - p_{ini}\right)\exp\left(\frac{\delta}{L^*}\right)\operatorname{erfc}\left(\sqrt{\frac{\delta}{L^*}}\right), \tag{9}$$

where

$$L^* = \frac{4}{f^2}\left(\frac{\rho c}{\Lambda}\right)^2\frac{\left(\sqrt{\alpha_{hy}}+\sqrt{\alpha_{th}}\right)^2}{V}. \tag{10}$$

No analytical solution exists for the Gaussian shear zone that is used in TPV105/106. However, the code can be run at high resolution to provide a reference solution. For TPV105/106 parameters, a solution to the slip-on-a-plane model with about 0.1% L2 (root-mean-square) error is obtained using $N = 100$ and $\Delta z = 1$ mm; with $N = 25$ and $\Delta z = 4$ mm, the L2 error is about 1%. The solutions with a finite shear zone are about an order of magnitude more accurate than those for slip-on-a-plane.