Summary

Dynamic seismic rupture requires accurate solutions nearby the time-varying boundary where shear stress jumps towards friction stress. Finite volume method for solving time domain elastodynamic system may provide this accuracy, especially on curved boundaries. We have developped a conservative scheme for crack problems. Numerical boundary conditions on an arbitray crack surface are specified following a discrete energy conservation rule.

Dynamic crack problem

Elastodynamic equations: velocity-stress formulation

$$\rho \frac{\partial \vec{v}}{\partial t} = \overrightarrow{div\, \underline{\sigma}} \qquad (1)$$

$$\frac{\partial \underline{\sigma}}{\partial t} = \lambda\, div\, \vec{v}\, I_n + \mu \left(\vec{\nabla}\, \vec{v} + (\vec{\nabla}\, \vec{v})^t \right) \qquad (2)$$

n is the space dimension
$\underline{\sigma}$ is the symmetric stress tensor
$\vec{v} \in \mathbb{R}^n$ is the velocity vector
ρ is the local density, and λ and μ are the local Lamé coefficients

Boundary crack condition (in plane fracture mode)

$$\vec{t}\ \underline{\sigma}(t,x)\ \vec{n} = g \quad \forall x \in \Gamma \qquad (3)$$

$\Gamma(\vec{x},t)$ is the crack surface
$\vec{n}$ and $\vec{t}$ are respectively the normal and tangential vectors to Γ
g is a bounded function, which may depend on time and friction parameters

Conservative law form

$$\Lambda\, \partial_t W - div\, \mathcal{F}(W) = 0 \qquad (4)$$

Λ is a diagonal matrix, depending on the density and the Lamé coefficients

Finite volume method

- Integration over each cell T_i ($\Omega = \cup T_i$)

$$\int_{T_i} \Lambda_{T_i} \partial_t W = \int_{\partial T_i} \mathcal{F}(W)\, \vec{n}\, dS \qquad (5)$$

- Finite volume (or P_0 DG) approach: discrete solution is assumed constant on each cell as well as material properties (figure 1).

$$\mathcal{A}_i\, \Lambda_{T_i} (\partial_t W)_{T_i} = \sum_{T_j \in V(T_i)} \Phi_{T_i,T_j} \qquad (6)$$

∘ $\mathcal{A}_i$ is the volume of T_i
∘ Φ_{T_i,T_j} is the flux integral across the interface $T_{ij} = T_i \cap T_j$

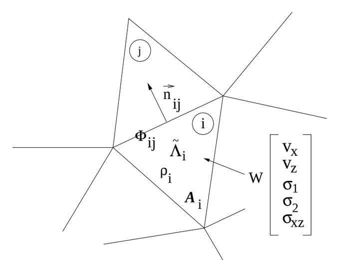

Fig 1. Two contiguous cells within the mesh. Φ_{ij} denotes the fluxes between T_i and T_j

Numerical flux and time integration scheme

Centered space scheme

Over $T_{ij} = T_i \cap T_j$,

$$\vec{v} \longrightarrow \frac{\vec{v}_i + \vec{v}_j}{2} \quad \text{and} \quad \vec{\sigma} \longrightarrow \frac{\vec{\sigma}_i + \vec{\sigma}_j}{2}$$

Leap frog time scheme

$$\begin{cases} \mathcal{A}_i\, \rho_i \dfrac{v_i^{n+\frac{1}{2}} - v_i^{n-\frac{1}{2}}}{\Delta t} = \displaystyle\sum_{j \in V(i)} \mathbb{N}_{ij} \dfrac{\sigma_i^n + \sigma_j^n}{2} \\ \mathcal{A}_i\, \bar{\Lambda}_i \dfrac{\sigma_i^{n+1} - \sigma_i^n}{\Delta t} = \displaystyle\sum_{j \in V(i)} \mathbb{N}_{ij}^t \dfrac{v_i^{n+\frac{1}{2}} + v_j^{n+\frac{1}{2}}}{2} \end{cases}$$

- Unstructured mesh: easy possible local refinement
- No staggered grid: all unknown variables and material parameters belong to the same cell
- Second-order time integration scheme

Energy conservation

- Energy in the continum

$$E = \underbrace{\int_\Omega \frac{1}{2} \rho \parallel \vec{v} \parallel^2}_{kinetic} + \underbrace{\int_\Omega \frac{1}{2}{}^t\vec{\sigma}\, \bar{\Lambda}\, \vec{\sigma}}_{mechanical}$$

- Discrete energy

$$\mathcal{E}^n = \frac{1}{2} \sum_{i \in V(\Omega)} \mathcal{A}_i \left(\rho_i\, (v_i^{n+\frac{1}{2}})^t\, v_i^{n-\frac{1}{2}} + (\sigma_i^n)^t\, \bar{\Lambda}_i\, \sigma_i^n \right)$$

- Discrete energy variation

$$\Delta\, \mathcal{E} := \mathcal{E}^{n+1} - \mathcal{E}^n = \sum_{interfaces}^{boundary} [\![v^{tang}\, \sigma^{normal} + v^{normal}\, \sigma^{tang}]\!]$$

where $[\![\, \xi\,]\!] = \xi^+ n^+ + \xi^- n^-$

- When no shear stress is applied, the energy over the domain must be constant

$$\Delta\, \mathcal{E} = 0$$

$\Longrightarrow$ This allows

∘ An unambiguous specification of the boundary crack conditions leading to stable solutions.
∘ A single way to construct an accurate solution for fracture mode II by disconnecting the cells on both sides of the interface defining an edge of the seismic rupture (figure 2). On one-side cell (T_i for instance)

∗ Tangential velocity is discontinuous: $v^{tang} \longrightarrow v_i^{tang}$

∗ Normal velocity is continuous: $v^{normal} \longrightarrow \dfrac{v_i^{normal} + v_j^{normal}}{2}$

∗ Energy conservation $\Longrightarrow \begin{cases} \sigma^{tang} \longrightarrow g \\ \sigma^{normal} \longrightarrow \dfrac{\sigma_i^{normal} + \sigma_j^{normal}}{2} \end{cases}$

(same thing for the corresponding cell T_j by inverting indexes i and j sharing the same discontinuous edge)

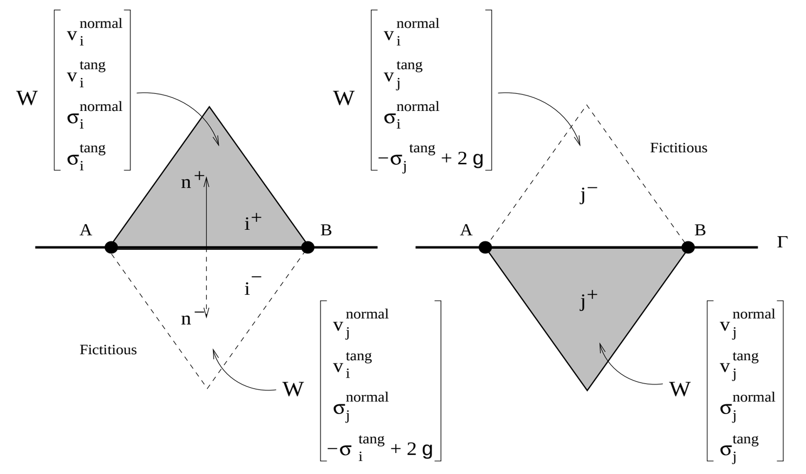

Fig 2. Two connected cells above and below an arbitrary interface AB of the crack surface

2D Numerical results

1. Spontaneous rupture propagation in heterogeneous medium

We have introduced a slip-weakening law in our rupture process in order to control the stress drop to the dynamic shear stress level (figure 3).

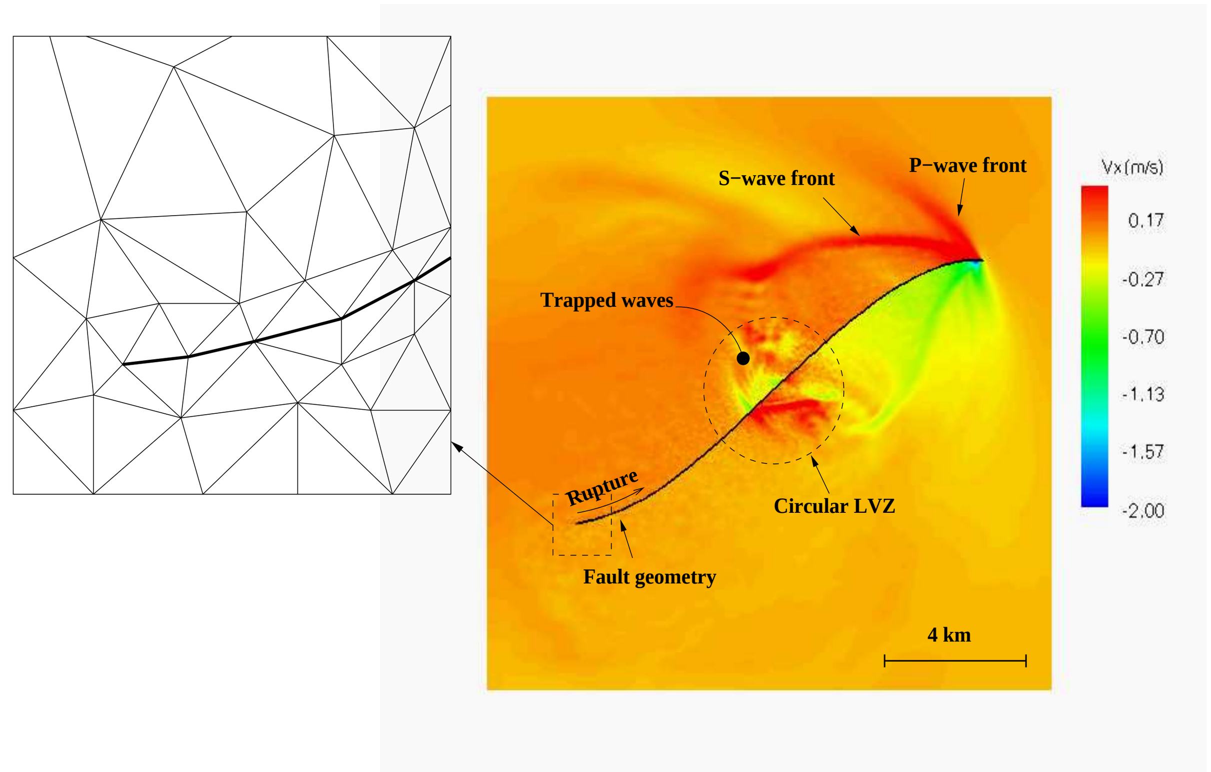

Fig 3. Snapshot of the horizontal velocity v_x for a spontaneous propagating rupture in heterogeneous medium. Spontaneous rupture propagating rightwards is governed by a slip weakening friction law

- Unstructured triangular meshes allow to mimic well the complex fault's geometry where high local density of triangles may guarantee the accuracy of the solution. Time step is still controlled by this smallest triangle.
- One portion of the fault coincides with the mesh segment: therefore, the fault has no numerical thickness.
- Strong effect of the low velocity zone (LVZ) on both radiated waves and rupture time evolution.

2. Mesh influence

Numerical solutions depend on the mesh definition and numerical investigations are performed in order to appreciate stability and accuracy of the solutions.

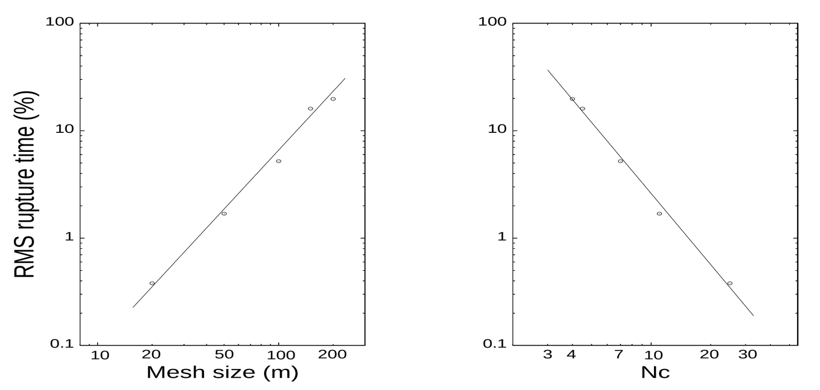

Fig 4. Difference in time of rupture as a function of the mesh size (left) and the number of mesh segments inside the cohesive zone (right), relative to a reference solution

Conclusion

- A new efficient finite volume method to simulate the spontaneous growth of an in-plane shear crack is presented
- Unstructured triangular meshes allow to mimic well any complex fault's geometry and also to refine locally discretisation. Still time integration is controlled by the smallest triangle.
- A suitable expression of discrete energy defines the appropriate numerical fracture boundary conditions to be imposed on the crack surface.
- Extension to high-order accuracy with discontinuous Galerkin discretization is possible.

3D Numerical results

1. Infinite Self-semilar dynamic rupture growth

For a self-semilar crack evolution at a prescribed velocity rupture and with an abrupt stress drop, we have compared numerical and analytical solutions.

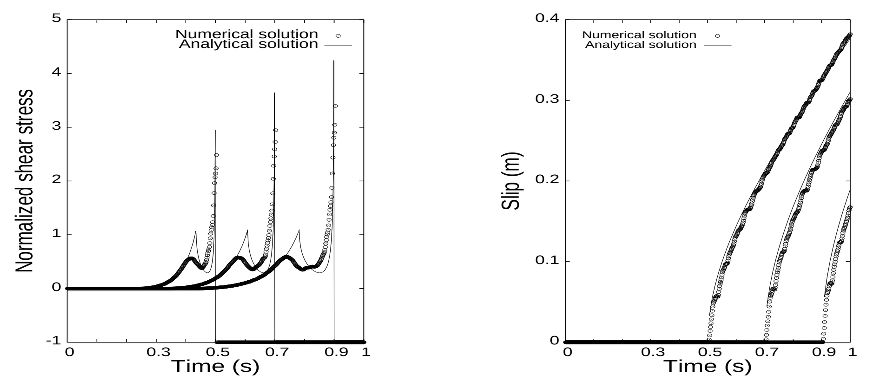

Fig 4. Comparison of the numerical (circles) and analytical (solid lines) solutions for the shear stress flux integral (right) and the fault slip (left)

- local mesh refinement on fault surface is necessary, otherwise singularity may disappear and spontaneous rupture propagation will be inaccurate.

2. Circular Self-semilar dynamic rupture growth

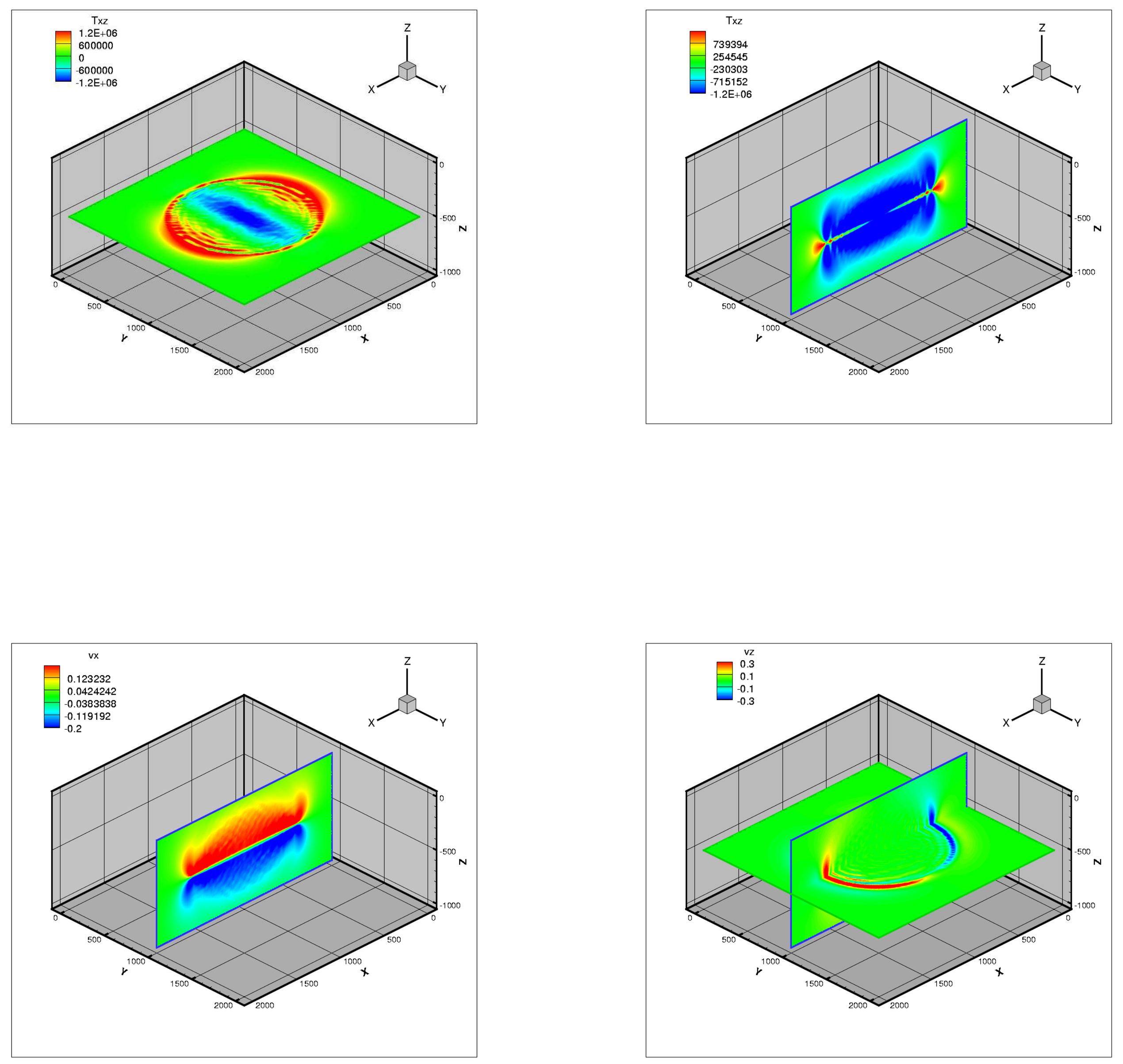

Fig 5. Snapshots of a circular self-similar crack propagation. Rupture occurs on the middle and propagates on radial direction with a prescribed velocity. On the strike plane, the shear stress τ_{xz} is set to be equal to the dynamic level, while τ_{yz} is equal zero. The top panels represent the shear stress τ_{xz} and the bottom panels represent the tangential velocity components v_x (left) and v_z (right)